On the Derivation of the L-to-Π Circuit Transformation Expressions

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1 Introduction

This article presents the derivation of the L-to-Π circuit transformation expressions. These formulas are used to convert the Network shown in Figure 1 to the Network shown in Figure 2. Examples are presented following the derivation.

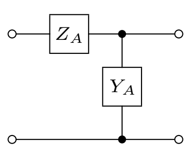

Figure 1: Network A

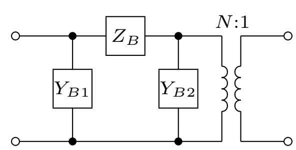

Figure 2: Network B

2 Derivation of Transformation Expressions

One of the simplest ways to develop equations for circuit transforms is to compare the ABCD (or Transfer) matricies for each network. If you are not familiar with ABCD parameters, or wish to refresh your memory, refer to [1] which has an excellent treatment of the subject.

Lets start by devloping the ABCD matrix for Network A in Figure 1. This is simply the product of the ABCD matrix for Z_A and Y_A and is given as

$$\begin{bmatrix} 1 & Z_A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_A & 1 \end{bmatrix} = \begin{bmatrix} 1 + Z_A Y_A & Z_A \\ Y_A & 1 \end{bmatrix} \tag{1}$$

The ABCD matrix for Network B in Figure 2 is the product of the ABCD matrix for Y_{B1}, Z_B, Y_{B2} and the transformer and is given as

$$
\begin{bmatrix} 1 & 0 \\ Y_{B1} & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_{B2} & 1 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & 1/N \end{bmatrix} = \begin{bmatrix} N(1+Z_B Y_{B2}) & Z_B/N \\ N(Y_{B1}+Y_{B1}Y_{B2}Z_B+Y_{B2}) & (Y_{B1}ZB+1)/N \end{bmatrix} \quad (2)
$$

Equating the ABCD matricies for Network A and Network B one obtains

$$
1 + Z_A Y_A = N(1 + Z_B Y_{B2}) \quad (3)
$$

$$
Z_A = Z_B/N \Rightarrow Z_B = Z_A N \quad (4)
$$

$$
Y_A = N(Y_{B1} + Y_{B1}Y_{B2}Z_B + Y_{B2}) \quad (5)
$$

$$
1 = (Y_{B1}Z_B + 1)/N \quad (6)
$$

Substituting (4) into (3) and rearanging for Y_{B2} gives

$$
Y_{B2} = \frac{1 + Z_A Y_A - N}{Z_A N^2} \quad (7)
$$

Substituting (4) into (6) and rearanging for Y_{B1} gives

$$
Y_{B1} = \frac{N-1}{Z_A N} \quad (8)
$$

Equations (4), (7) and (8) allow one to determine Z_B, Y_{B2} and Y_{B1} from Z_A, Y_A and the transformer turns ratio N.

3 Determining Element Values

Now that the general transformation expressions have been dervided, two specific cases will be studied. The first case occurs when Z_A, Y_A, Y_{B1}, Z_B and Y_{B2} are inductive reactances and the user specifies N. The second case occurs when Z_A, Y_A, Y_{B1}, Z_B and Y_{B2} are capacitive reactances and the user specifies N.

3.1 Case 1: Inductive Elements

Case 1 is shown in Figure 3.

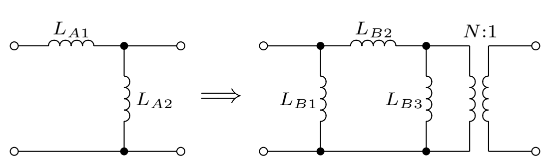

Figure 3: Transformation of inductive L network.

For this Case:

- Z_A is an inductor of impedance sL_{A1}.
- Y_A is an inductor of admittance $1/sL_{A2}$.
- Y_{B1} is an inductor of admittance $1/sL_{B1}$.
- Z_B is an inductor of impedance sL_{B2}.
- Y_{B2} is an inductor of admittance $1/sL_{B2}$.

Where s is the complex frequency variable.

To derive the specific formulas relating Z_A and Y_A to Y_{B1}, Z_B and Y_{B2} we must substitute the above reactances into (4), (7) and (8). Substituting the reactances into (4) gives

$$\begin{aligned} sL_{B2} &= sL_{A1}N \\ \Rightarrow L_{B2} &= NL_{A1} \end{aligned} \tag{9}$$

Substituting the reactances into (8) gives

$$\begin{aligned} \frac{1}{sL_{B1}} &= \frac{N-1}{sL_{A1}N} \\ \Rightarrow L_{B1} &= \frac{L_{A1}N}{N-1} \end{aligned} \tag{10}$$

and finally substituting the reactances into (7) gives

$$\begin{aligned} \frac{1}{sL_{B3}} &= \frac{1+\frac{sL_{A1}}{sL_{A2}}-N}{sL_{A1}N^2} \\ \Rightarrow L_{B3} &= \frac{L_{A1}N^2}{1+\frac{L_{A1}}{L_{A2}}-N} \end{aligned} \tag{11}$$

A summary of the results is shown below. Note that $N > 1$ to achieve positive element values.

$$\boxed{\begin{aligned} L_{B1} &= \frac{L_{A1}N}{N-1} \\ L_{B2} &= NL_{A1} \\ L_{B3} &= \frac{L_{A1}N^2}{1+\frac{L_{A1}}{L_{A2}}-N} \end{aligned}}$$

To make L_{B1} and L_{B3} equal make:

$$N = \sqrt{1+\frac{L_{A1}}{L_{A2}}} \tag{12}$$

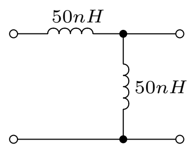

Figure 4: Inductive L-network.

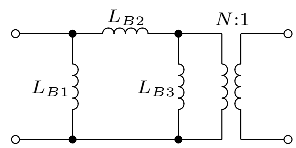

Figure 5: Transformed inductive network.

Lets look at an example to illustrate the use of this transformation. Suppose we wish to transform the network in Figure 4 to that shown in Figure 5.

First we chose a value for the transformer turns ratio N. We will arbitrarily chose a value of 1.25. We then use the equations shown above to calculate $LB1$, $LB2$ and $LB3$.

$$\begin{aligned} L_{B1} &= \frac{50nH \times 1.25}{1.25 - 1} \\ &= 250nH \end{aligned} \tag{13}$$

$$\begin{aligned} L_{B2} &= 1.25 \times 50nH \\ &= 62.5nH \end{aligned} \tag{14}$$

$$\begin{aligned} L_{B3} &= \frac{50nH \times 1.25^2}{1 + \frac{50nH}{50nH} - 1.25} \\ &= 104.2nH \end{aligned} \tag{15}$$

3.2 Case 2: Capacitive Elements

Case 2 is shown in Figure 6.

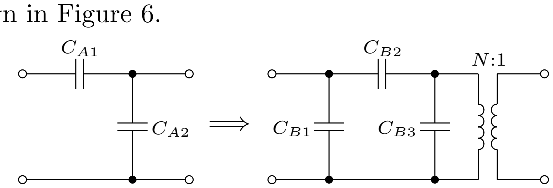

Figure 6: Transformation of capacitive L network.

For this Case:

- Z_A is an inductor of impedance $1/sC_{A1}$.
- Y_A is an inductor of admittance sC_{A2}.
- Y_{B1} is an inductor of admittance sC_{B1}.

- Z_B is an inductor of impedance $1/sC_{B2}$.
- Y_{B2} is an inductor of admittance sC_{B3}.

To derive the specific formulas relating Z_A and Y_A to Y_{B1}, Z_B and Y_{B2} we must substitute the above reactances into (4), (7) and (8). Substituting the reactances into (4) gives

$$\begin{aligned} \frac{1}{sC_{B2}} &= \frac{N}{sC_{A1}} \\ \Rightarrow C_{B2} &= \frac{C_{A1}}{N} \end{aligned} \tag{16}$$

Substituting the reactances into (8) gives

$$\begin{aligned} sC_{B1} &= \frac{N-1}{\frac{N}{sC_{A1}}} \\ \Rightarrow C_{B1} &= \frac{C_{A1}(N-1)}{N} \end{aligned} \tag{17}$$

and finally substituting the reactances into (7) gives

$$\begin{aligned} sC_{B3} &= \frac{1+\frac{sC_{A2}}{sC_{A1}}}{\frac{N^2}{sC_{A1}}} - \frac{sC_{A1}}{N} \\ \Rightarrow C_{B3} &= \frac{C_{A1}(1-N)+C_{A2}}{N^2} \end{aligned} \tag{18}$$

A summary of the results is shown below. As with the Case 1, $N > 1$ to achieve positive element values.

$$\boxed{\begin{aligned} C_{B1} &= \frac{C_{A1}(N-1)}{N} \\ C_{B2} &= \frac{C_{A1}}{N} \\ C_{B3} &= \frac{C_{A1}(1-N)+C_{A2}}{N^2} \end{aligned}}$$

To make C_{B1} and C_{B3} equal make [2]:

$$N = \sqrt{1+\frac{C_{A2}}{C_{A1}}} \tag{19}$$

Lets look at an example to illustrate the use of this transformation. Suppose we wish to transform the network in Figure 7 to that shown in Figure 8.

First we chose a value for the transformer turns ratio N. We will arbitrarily chose a value of 1.25. We then use the equations shown above to calculate $CB1$, $CB2$ and $CB3$.

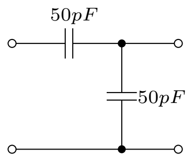

Figure 7: Capacitive L-network.

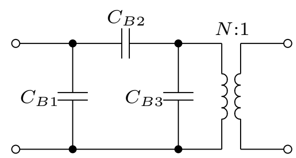

Figure 8: Transformed capacitive network.

$$C_{B1} = \frac{50pF(1.25-1)}{1.25} = 10pF \tag{20}$$

$$C_{B2} = \frac{50pF}{1.25} = 40pF \tag{21}$$

$$C_{B3} = \frac{50pF(1-1.25)+50pF}{1.25^2} = 24pF \tag{22}$$

4 Conclusion

Formulas have been derived to transform an L network to a $\prod$ network and exaples show how these equations can used with inductive and capacitive networks. It has been shown that ABCD parameters are convenient for deriving such transformation expressions.

References

[1] David M. Pozar. *Microwave Engineering*. Wiley Text Books, second edition, 1997. ISBN 0471170968.

[2] Brian J. Minnis. Classes of sub-miniature microwave printed circuit filters with arbitrary passband and stopband widths. *IEEE Trans. Microwave Theory Tech.*, MTT-30(11):p. 1900, Nov. 1982.